MOTION OF A MAGNETIZED EQUATORIAL SATELLITE ABOUT ITS CENTER OF MASS IN A CIRCULAR ORBIT WITH INTERACTION OF THE EARTH'S MAGNETIC AND GRAVITATIONAL FIELDS

PMM Vol. 31, No. 5, 1967, pp. 947-950

A. A. KHENTOV
(Moscow)

(Date received not given)

The Earth's magnetic field can have a very marked effect on the motion of a magnetized space vehicle about its center of mass. Several papers (e.g. [1 to 3]) analyzing the possibility of passive stabilization of such a satellite along a line of force of the geomagnetic field have already appeared.

The problems considered in the present paper are closely related to the classical problems of rotation of a solid body about a stationary point in homogeneous and Newtonian force fields.

1. The equations of motion. Let us introduce the absolute right geocentric coordinate system $X_1 X_2 X_3$ (see Fig. 1) whose axis X_3 coincides with the Earth's axis of rotation. At the center of mass G of the satellite we take three additional right frames $x_1 x_2 x_3$, $y_1 y_2 y_3$ and $z_1 z_2 z_3$. The axes x_i are parallel to the respective axes X_i ; y_j are the principal central inertial axes of the satellite. The axis z_3 of the orbital coordinate system z_k coincides with the radius vector of the center mass of the satellite ; z_1 coincides with the orbit transversal ($i, j, k = 1, 2, 3$).

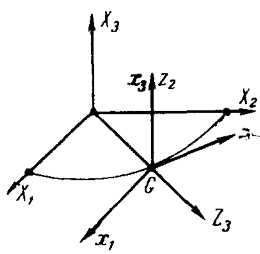

Fig. 1

The position of the trihedron y_j relative to the axes x_i and z_k will be specified by means of the direction cosines γ_{ji} and α_{jk}, respectively ; clearly, in this case $\gamma_{13} = \alpha_{12}$, $\gamma_{23} = \alpha_{22}$ and $\gamma_{33} = \alpha_{32}$.

Let us adopt the dipole model of the Earth's magnetic field. Then, assuming the coincidence of the geographic and geomagnetic poles, for the magnetic intensity vector of this field we have (in accordance with [4]) $\mathbf{H} = \mu_e \mathbf{n}/R^3$, where μ_e is the geomagnetic constant, R is the radius vector of the center of mass of the satellite, and $\mathbf{n}$ is the unit vector of the axis x_3.

We shall assume that the magnetic moment $\mathbf{I}°$ of the satellite consists of the constant component $\mathbf{I}$ and the magnetic moment of the shell. The direction cosines of the vector relative to the thrihedron y_j will be denoted by η_j.

The satellite shell will be assumed to be geometrically symmetrical and magnetized along its axis of symmetry. We also assume that the axis of symmetry of the shell coincides with one of the principal inertial axes of the satellite, e.g. y_3. The magnetic moment of the shell in the first approximation can be written (according to [5 and 6]) as $\zeta\gamma_{33} / H$, where $\zeta = (\mu_0 - 1)\, v\, H^2 / 4\pi$, and where μ_0 and v are the magnetic permeability of the shell and the shell volume.

In a magnetic field of intensity **H** such a satellite is acted on by the magnetic moment $\mathbf{M} = \mathbf{I}^\circ \times \mathbf{H}$. The gravitational moment can be written as in [2]. The other moments will be neglected. The orbit will be considered given. The equations of motion of the satellite about its center of mass in this bounded problem are as follows:

$$A_1 dp_1 / dt + (A_3 - A_2)\, p_2 p_3 = IH\,(\eta_2\gamma_{33} - \eta_3\gamma_{23}) - \zeta\gamma_{23}\gamma_{33} + 3\delta\,(A_3 - A_2)\alpha_{23}\alpha_{33} \quad (1.1)$$
$$A_2 dp_2 / dt + (A_1 - A_3)\, p_1 p_3 = IH\,(\eta_3\gamma_{13} - \eta_1\gamma_{33}) + \zeta\gamma_{13}\gamma_{33} + 3\delta\,(A_1 - A_3)\,\alpha_{13}\alpha_{33}$$
$$A_3 dp_3/dt + (A_2 - A_1)\, p_1 p_2 = IH\,(\eta_1\gamma_{23} - \eta_2\gamma_{13}) + 3\delta\,(A_2 - A_1)\alpha_{13}\alpha_{23}$$

Here A_1, A_2, A_3 are the moments of inertia of the satellite relative to the axes y_1, y_2, y_3, respectively; p_1, p_2, p_3 are the projections of the absolute angular velocity of the satellite on the same axes; μ is the gravitational constant; $\delta = \mu / R^3$.

Let us complement system (1.1) with Poisson relations for the direction cosines

$$d\gamma_{13} / dt = \gamma_{23} p_3 - \gamma_{33} p_2 \quad (123) \qquad (1.2)$$
$$d\alpha_{13} / dt = \alpha_{23} p_3 - \alpha_{33} p_2 + \omega\alpha_{11} \quad (123) \qquad (1.3)$$
$$d\alpha_{11} / dt = \alpha_{21} p_3 - \alpha_{31} p_2 - \omega\alpha_{13} \quad (123) \qquad (1.4)$$

Here the symbol (123) indicates that the remaining relations can be obtained by cyclic permutation; ω is the angular velocity of rotation of the satellite center of mass along its orbit.

If the orbit is circular, then system (1.1) to (1.4) admits of the existence of a Jacobi-type integral

$$A_1 p_1^2 + A_2 p_2^2 + A_3 p_3^2 - 2\mathbf{I}\cdot\mathbf{H} - \zeta\ \gamma_{23}^2 + 3\omega^2\,(A_1\alpha_{13}^2 + A_2\alpha_{23}^2 + A_3\alpha_{33}^2) - 2\omega\,(A_1 p_1\gamma_{13} + A_2 p_2\gamma_{23} + A_3 p_3\gamma_{33}) = h \qquad (1.5)$$

This integral can be given a clear physical interpretation by converting to the angular velocities q_1, q_2, q_3 relative to the orbital axes. We obtain

$$A_1 q_1^2 + A_2 q_2^2 + A_3 q_3^2 - 2\mathbf{I}\cdot\mathbf{H} - \zeta\,\gamma_{23}^2 + 3\omega^2\,(A_1\alpha_{13}^2 + A_2\alpha_{23}^2 + A_3\alpha_{33}^2) - \omega^2\,(A_1\gamma_{13}^2 + A_2\gamma_{23}^2 + A_3\gamma_{33}^2) = h \qquad (1.6)$$

This relation yields the energy conservation law in the form $T + V_1 + V_2 + V_3 = h$, where T is the kinetic energy in relative motion, V_1 is the potential energy of the magnetic forces, V_2 is the potential energy of the gravitational forces, and V_3 is the potential energy of the centrifugal forces.

The above system appears not to have other integrals.

2. Rotation of a satellite in the magnetic field. With certain satellites the magnetic moments can play the major role, the influence of the gravitational field being limited to the production of perturbations. In view of this it is interesting to investigate the motion of a satellite in a magnetic force field alone. In the absence of gravitational moments we can write out yet another first integral which reflects the constancy of the projection of the satellite's kinetic moment relative to its center of mass on the axis x_3. This integral is of the form

$$A_1 p_1\gamma_{13} + A_2 p_2\gamma_{23} + A_3 p_3\gamma_{33} = K \qquad (2.1)$$

Let us rewrite (1.5) and (2.1), introducing in the usual way the Euler angles of nutation θ, precession ψ, and intrinsic rotation φ in order to define the position of the satellite relative to the axes x_1. We obtain

$$A_1 (\psi^{\cdot} \sin\theta \sin\varphi + \theta^{\cdot} \cos\varphi)^2 + A_2 (\psi^{\cdot} \sin\theta \cos\varphi - \theta^{\cdot} \sin\varphi)^2 + A_3(\psi^{\cdot} \cos\theta + \varphi^{\cdot})^2$$
$$- 2IH (\eta_1 \sin\theta \sin\varphi + \eta_2 \sin\theta \cos\varphi + \eta_3 \cos\theta) - \zeta \cos^2\theta = h$$
$$A_1 (\psi^{\cdot} \sin\theta \sin\varphi + \theta^{\cdot} \cos\varphi) \sin\theta \sin\varphi + A_2 (\psi^{\cdot} \sin\theta \cos\varphi -$$
$$- \theta^{\cdot} \sin\varphi) \sin\theta \cos\varphi + A_3 (\psi^{\cdot} \cos\theta + \varphi^{\cdot}) \cos\theta = K$$

The coordinate ψ is cyclic. Let us consider the steady-state rotations of the satellite defined by the conditions

$$\varphi^{\cdot} = \theta^{\cdot} = 0, \quad \varphi = \varphi_0, \quad \theta = \theta_0, \quad \psi^{\cdot} = \psi_0^{\cdot} \tag{2.2}$$

To investigate these rotations we apply Routh's theorem [7 and 8]. The Routh potential Π for the scleronomic system under consideration is

$$\Pi = K_0^2 / 2a - IH (\eta_1 \sin\theta \sin\varphi + \eta_2 \sin\theta \cos\varphi + \eta_3 \cos\theta) - \zeta \cos^2\theta / 2$$
$$a = A_1 \sin^2\theta \sin^2\varphi + A_2 \sin^2\theta \cos^2\varphi + A_3 \cos^2\theta$$

Here K_0 represents the integral of the kinetic moment computed under conditions (2.2). The angles φ_0 and θ_0 must be determined from Expressions $\partial\Pi/\partial\varphi = 0$ and $\partial\Pi/\partial\theta = 0$ which are of the form

$$[(A_1 - A_2) \psi_0^{\cdot 2} \sin\theta \sin 2\varphi + 2IH (\eta_1 \cos\varphi - \eta_2 \sin\varphi)] \sin\theta = 0$$
$$(b\psi_0^{\cdot 2} - \zeta) \sin 2\theta + 2IH [(\eta_1 \sin\varphi + \eta_2 \cos\varphi) \cos\theta - \eta_3 \sin\theta] = 0 \tag{2.3}$$
$$b = A_1 \sin^2\varphi + A_2 \cos^2\varphi - A_3$$

These equations isolate the permanent axes in the body of the satellite. In order for the rotations about these axes to be stable the Routh potential must have a minimum. The sufficient conditions for stability are

$$(A_1 - A_2) \psi_0^{\cdot 2} \sin^2\theta (-\cos 2\varphi + B_1 \sin^2\theta \sin^2 2\varphi) + IH \sin\theta (\eta_1 \sin\varphi + \eta_2 \cos\varphi) > 0$$
$$[(A_1 - A_2) (-\cos 2\varphi + B_1 \sin^2\theta \sin^2 2\varphi) \psi_0^{\cdot 2} \sin^2\theta + IH \sin\theta (\eta_1 \sin\varphi +$$
$$+ \eta_2 \cos\varphi)] [b\psi_0^{\cdot 2} (-\cos 2\theta + b_1 \sin^2 2\theta) + \zeta \cos 2\theta + IH (\eta_1 \sin\theta \sin\varphi +$$
$$+ \eta_2 \sin\theta \cos\varphi + \eta_3 \cos\theta)] + [B_2 \psi_0^{\cdot 2} (1 - 2b_1 \sin^2\theta) \sin 2\theta \sin 2\varphi + IH \cos\theta \times$$
$$\times (\eta_1 \cos\varphi - \eta_2 \sin\varphi)]^2 > 0$$
$$B_1 = (A_1 - A_2) / a, \qquad B_2 = (A_1 - A_2) / 2, \qquad b_1 = b / a$$

The motion of a dynamically symmetrical satellite ($A_1 = A_2 \neq A_3$) about its center of mass is equivalent to the motion of a dynamically symmetrical solid about a stationary point in a Newtonian central force field when the distance to the center of attraction substantially exceeds the dimensions of the body. Such a problem admits of complete integration [2].

If the magnetic moment of the satellite shell is small as compared with the magnetic moment I, then the motion of the satellite about its center of mass closely resembles the motion of a heavy solid about a fixed point.

In fact, if we neglect the magnetization of the shell, then the equations of rotation of the satellite coincide with the equations of rotation of a heavy solid to within the notation.

A case analogous to the Euler case results if I is equal to zero. A dynamically symmetrical satellite with the magnetic moment directed along its axis of dynamic symmetry moves in the same way as a heavy solid in the Lagrange-Poisson case; on the other hand, if the magnetic moment lies in the equatorial plane of the central ellipsoid of inertia of the satellite and if the condition $A_1 = A_2 = 2A_3$ is fulfilled, then the motion of such a satellite is analogous to the motion of a heavy solid in the Kovalevski case.

The steady-state rotations of a heavy solid are considered in detail in [9 to 11].

In the case of a constantly magnetized satellite the equation of the cone analogous to the Staude cone is of the form

$$(A_2 - A_3)\,\eta_1\gamma_{23}\gamma_{33} + (A_3 - A_1)\,\eta_2\gamma_{13}\gamma_{33} + (A_1 - A_2)\,\eta_3\gamma_{13}\gamma_{23} = 0$$

Let us consider, for example, the following particular solution of system (1.1),(1.2):

$$p_j = 0, \qquad \gamma_{j3} = \eta_j$$

This particular solution is associated with equilibrium of the satellite in absolute axes where its magnetic moment is colinear with the magnetic intensity vector (the analogous case for the motion of a sold is the equilibrium state in which the center of gravity occupies its lowest position). It is clear that such equilibrium must be stable.

Let us prove this with the aid of integral (1.5). We denote the angle between the magnetic moment vector **I** of the satellite and the vector **H** by ν and expand $\cos\nu$ in a Taylor series. Substituting the expansion into (1.5), we obtain

$$A_1 p_1^2 + A_2 p_2^2 + A_3 p_3^2 + IH\nu^2 + \Delta = h^\circ \qquad (h^\circ = h + 2\,IH)$$

Here Δ is the sum of terms of higher than the third order in ν. In this form the integral is a positively defined function of its arguments. Hence, the equilibrium state under consideration is, in fact, stable in p_j and ν.

It is clear that the other possible equilibrium state where the vectors **I** and **H** are antiparallel is unstable (the analogous case for a solid is the equilibrium state in which the center of mass occupies its uppermost position).

3. Motion of the satellite with interaction of the magnetic and gravitational fields. Let us assume that the vector **I** coincides with the axis y_3 of the satellite.

Let us investigate the possibility of the equilibrium of the satellite in absolute axes. As we infer from system (1.1), such equilibrium requires simultaneous fulfillment of the following Eqs. :

$$(IH + \zeta\gamma_{33})\,\gamma_{23} - 3\delta\,(A_3 - A_2)\,\alpha_{23}\alpha_{33} = 0$$

$$(IH + \zeta\gamma_{33})\,\gamma_{13} + 3\delta\,(A_1 - A_3)\,\alpha_{13}\alpha_{33} = 0$$

$$(A_2 - A_1)\,\alpha_{13}\alpha_{23} = 0$$

This means that absolute equilibrium is possible only if the axis y_3 is the axis of dynamic symmetry of the satellite and if this axis coincides with the normal to the orbital plane.

Let us now consider the precessions of the satellite for which it maintains an unchanged position relative to the orbital axes. From system (1.1) we obtain the necessary conditions for the existence of such a relative equilibrium (in a circular orbit),

$$\{IH + [\zeta + (A_3 - A_2)\,\omega^2]\,\alpha_{32}\}\,\alpha_{22} - 3\omega^2\,(A_3 - A_2)\,\alpha_{23}\alpha_{33} = 0$$

$$\{IH + [\zeta - (A_1 - A_3)\,\omega^2]\,\alpha_{32}\}\,\alpha_{12} + 3\omega^2\,(A_1 - A_3)\,\alpha_{13}\alpha_{33} = 0$$

$$(A_2 - A_1)\,(-\alpha_{12}\alpha_{23} + 3\,\alpha_{13}\alpha_{23}) = 0$$

Let us note down several of the possible precession states.

The state $\alpha_{13} = \pm 1, \alpha_{33} = \pm 1$. The axis y_3 is vertical.

The state $\alpha_{12} = \alpha_{13} = 0$. The axis y_3 lies in the plane z_2z_3 and the axis y_1 is normal to this plane. Here $\cos\nu = -IH \,/\, [4\omega^2\,(A_3 - A_2) + \zeta]$

The state $\alpha_{12} = \alpha_{23} = 0$. The axis y_3 lies in the plane z_1z_3 and the axis y_1 is normal to this plane. Here

$$\cos\nu = -IH / [4\omega^2(A_3 - A_2) + \zeta]$$

The state $\alpha_{22} = \alpha_{13} = 0$. The axis y_3 lies in the plane $z_1 z_2$ and the axis y_2 is normal to this plane. Here $\cos\nu = -IH / [\omega^2(A_3 - A_1) + \zeta]$

The state $\alpha_{22} = \alpha_{23} = 0$. The axis y_3 lies in the plane $z_2 z_3$ and the axis y_2 is normal to this plane. Here $\cos\nu = -IH / [4\omega^2(A_3 - A_1) + \zeta]$

Let us investigate the stability of the relative equilibrium of the satellite in which its magnetic moment is colinear with the magnetic intensity vector (the state $\alpha_{13} = 1$).

Let us make use of integral (1.6) written as

$$A_1 q_1^2 + A_2 q_2^2 + A_3 q_3^2 - 2IH\gamma_{33} + \zeta(\gamma_{13}^2 + \gamma_{23}^2) + 3\omega^2[(A_2 - A_1)\alpha_{23}^2 + (A_3 - A_1)\alpha_{33}^2] + \omega^2[(A_3 - A_1)\gamma_{13}^2 + (A_3 - A_2)\gamma_{23}^2] = h_1$$

Expanding γ_{33} in a Taylor series, we find that this equilibrium is stable if $A_3 > A_2 > A_1$. Thus, if the axis y_3 of the satellite is normal to the orbital plane and if it is associated with the largest of the moments of inertia, then the conditions of stability of the relative equilibrium are of the same form as in the absence of a magnetic field [2]. The moments of the magnetic forces improve practical stability (the condition $A_3 > A_2$ can be replaced by the weaker condition $\zeta + \omega^2(A_3 - A_2) > 0$).

The regular precession states of a dynamically symmetrical satellite in a gravitational field are considered in [12].

The author is grateful to V. V. Beletskii for his interest in the present study.

BIBLIOGRAPHY

1. Fischell, R. E., Passive magnetic attitude control for Earth satellites. 8th Annual Meeting of the Americal Astronautical Society, 1962.
2. Beletskii, V. V., Motion of an Artificial Satellite about its Center of Mass, "Nauka", Moscow, 1965.
3. Khentov, A. A., Passive stabilization of artificial satellites relative to the geomagnetic field. Kosmicheskie Issledovaniia No. 4, 1967.
4. Landau, L. D. and Lifshits, E. M., Theoretical Physics, Vol. 2, Field Theory, 4th ed., Fizmatgiz, Moscow, 1962.
5. Colombo, G., On the motion of Explorer XI around its center of mass. Perp. Am. Astr. Soc., No. 45, 1962.
6. Hagihara, Y., Rotation of an Earth Satellite in Flight along its Orbit. Smithsonian Contr., Astr. Vol. 5, No. 9, 1961.
7. Gantmakher, F. R., Lectures on Analytical Mechanics. Fizmatgiz, Moscow, 1960.
8. Pozharitskii, G. K., On the construction of the Liapunov functions from the integrals of the equations for perturbed motion. PMM Vol. 22, No. 2, 1958.
9. Staude, O., Über permanente Rotationsachsen bei der Bewegung eines schweren Körpers um einen festen Punkt. J. für die reine und ang. Math. Vol. 113, 1894.
10. Grammel, R., The Gyroscope, its Theory and Applications. Vol. 1, Izd. Inostr. Lit., Moscow, 1952.
11. Rumiantsev, V. V., The stability of permanent rotations of a heavy solid. PMM Vol. 20, No. 1, 1956.
12. Chernous'ko, F. L., On the stability of regular precession of a satellite. PMM Vol. 28, No. 1, 1964.

Translated by A. Y.